Reflections on Orlov

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Looking Backwards and Forwards

Background: Relevant Logic

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- Material conditional: $A \rightarrow B := \neg A \vee B$
- *MEq*: $A, B \models A \leftrightarrow B$

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- $A \rightarrow B := \neg(A \circ \neg B)$

- $A + \neg A$

Orlov's Motivation

Orlov and Intuitionism

One can also assert that the development of the ... [logical theory that follows] is a problem waiting to be addressed, of the necessity to reconcile symbolic logic with new techniques introduced by intuitionism.

Orlov and Intuitionism

An important aspect of intuitionism lies in the fact that in works of intuitionists concepts used depend not directly on used propositions A, B, C, but on functions of the latter, of the following form: "A is established", "A is provable", "A leads to absurdity", "absurdity of A is absurd", etc. That intuitionists employ similar functions is a fact, and this fact cannot be ignored in the development of mathematical logic.

Orlov and Intuitionism

However, the introduction of similar functions into classical mathematical logic leads to contradiction with the law "tertium non datur": later it will be shown that it is not possible at all. *Meanwhile, the "calculus of joint compatibility of propositions" allows for the introduction of said functions and symbolic operations on them. As a result, the position of intuitionism does not require the rejection of the law "tertium non datur", that is, is in complete agreement with it.*

Intuitionism and Provability

Intuitionism and Provability

*We claim that similar functions [GP: to □ and □¬], even if without special symbolic denotation, in fact are introduced by intuitionists into mathematics. "For any given system, every condition [*свойство*] is either correct or impossible" is evaluated differently in purely formal mathematics and in works of intuitionists precisely because it is not understood in the same sense. If in the first case any condition is taken "in itself", then in the second case the question of the provability of the condition is advanced. In the first case the question about the existence of a condition is treated in the sense of the expression "A", while in the second, in the sense of "□A".*

Intuitionism Provability

Brouwer examines the case where, from a condition follows its negation, only as a particular case of "absurdity", understanding this expression more broadly. He uses the expression "absurd" in those cases where it may be proven that A contradicts some axiom or some proven sentence. But our expression $\Box\neg A$ also has the same meaning. From here Brouwer's claim becomes clearer, that the correctness of a condition and its absurdity form a complete disjunction only in a defined finite system. Even though the conditions of such a system can remain partially unknown, one needs only sufficient amount of time for their complete determination.

S4

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- $\vdash \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
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- Prove = establish as true
- $\times \quad \neg\Box A \rightarrow \Box\neg\Box A$

Orlov's Motivation

The introduction of the said functions [GP: □ and □¬] into classical mathematical logic is not possible, since the interpretation of the concept of "following from", as material implication, erases meaning for all the expressions proven for the functions we introduced. Besides, in classical theory there are sentences derived and proven, that from our point of view cannot be evaluated in any way other than false.

Orlov's Motivation

For example, from the sentence provable in classical theory "all true sentences are equivalent", the following conclusion follows:

$$A \leftrightarrow \Box A \leftrightarrow \Box\neg\Box A \leftrightarrow \Box\neg\Box\neg\Box A$$

Such a conclusion makes the introduction of this type of functions devoid of any kind of meaning; in this case when constructing schemata of transfinite inferences, there would be no other way except the rejection of the law "tertium non datur".

Došen's Comment

In the last paragraph of his paper, Orlov comes to a rather odd conclusion. He thinks that the modal postulates of S4 can be added to OR, but cannot be added to classical logic. Orlov's opinion is that his modal theorems become senseless if we interpret $\rightarrow$ as material implication, and he somehow infers from the classical principle "all true propositions are equivalent" that we cannot add the modal postulates of S4 to classical logic without also adding $A \rightarrow \Box A$, which would, of course, make our modal system trivial and useless.

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- S4 based on classical logic
- the material conditional (with *MEq*)
- Excluded Middle
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- Lewis:
  - $\Diamond(A \wedge B)$
  - $A \multimap B := \neg\Diamond(A \wedge \neg B)$

Orlov's Reasoning?

Either A or $\neg A$. Suppose that A. In an intuitionist context, this means that A is provable, that is, $\Box A$. Since both are true, $A \leftrightarrow \Box A$ by *MEq*. Suppose that A is false, $\neg A$. Again, in an intuitionist context, this means that A is refutable. That is, $\Box\neg A$. So $\neg\Box A$. Hence, $\Box A$ is false as well. Thus, again by *MEq*, $A \leftrightarrow \Box A$, in either case. All modal distinctions collapse.

Reasoning under Assumption

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- The validity of the following inference depends crucially on whether $\mathcal{P}$ has undischarged assumptions:

$$\frac{\begin{array}{c}\mathcal{P}\\ A\end{array}}{\Box A}$$

BCK Semantics

- A proof of $A \rightarrow B$ is a construction which converts a proof of A into a proof of B

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- $A \rightarrow \Box A$ appears to follow.

Kripke Models

- $w \Vdash \Box A$ iff $w \Vdash A$

Kripke Models

- $w \Vdash \Box A$ iff $w \Vdash A$
- The validity if $A \leftrightarrow \Box A$ is immediate.

Paraconsistent Logic

LP$\Box$

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  - R is a reflexive and transitive binary relation on W
  - $\delta(p) = \langle X, Y \rangle$, where $X \cup Y = W$
  - Write X and Y as $\delta^+(p)$ and $\delta^-(p)$

Truth/Falsity Conditions

- $w \Vdash^{+} p$ iff $w \in \delta^{+}(p)$
- $w \Vdash^{-} p$ iff $w \in \delta^{-}(p)$
- $w \Vdash^{+} A \vee B$ iff $w \Vdash^{+} A$ or $w \Vdash^{+} B$
- $w \Vdash^{-} A \vee B$ iff $w \Vdash^{-} A$ and $w \Vdash^{-} B$
- $w \Vdash^{+} A \wedge B$ iff $w \Vdash^{+} A$ and $w \Vdash^{+} B$
- $w \Vdash^{-} A \wedge B$ iff $w \Vdash^{-} A$ or $w \Vdash^{-} B$
- $w \Vdash^{+} \neg A$ iff $w \Vdash^{-} A$
- $w \Vdash^{-} \neg A$ iff $w \Vdash^{+} A$
- $w \Vdash^{+} \Box A$ iff for all $w' \in W$ such that wRw', $w' \Vdash^{+} A$
- $w \Vdash^{-} \Box A$ iff for some $w' \in W$ that wRw', $w' \Vdash^{-} A$

Validity

- $\Sigma \models A$ iff for every world, w, of every $\mathfrak{I}$, if $w \Vdash^+ B$ for all $B \in \Sigma$, $w \Vdash^+ A$.

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- $\times \quad \Box(A \supset B) \models \Box A \supset \Box B$
- if $A_1, ..., A_n \models B$ then $\Box A_1, ..., \Box A_n \models \Box B$

$\Box$ as a Provability Operator for a Theory

- Take an interpretation, $\mathfrak{I}$, and a world, w:
- $\Pi_w = \{A : w \Vdash^+ \Box A\}$

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- Π_w is a theory
  - For if $B_1, ..., B_n \in \Pi_w$, $w \Vdash^+ B_1, ..., B_n$
  - If $B_1, ..., B_n \models A$, $\Box B_1, ..., \Box B_n \models \Box A$
  - So $w \Vdash^+ \Box A$
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- $w \Vdash^+ \Box A$ iff $A \in \Pi_w$

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- $w \Vdash^+ \Box A$ iff $A \in \Pi_w$
- So $\Box A$ (in w) means provability in Π_w.

And...

$$\Pi_w \vdash A \quad \Rightarrow \quad w \Vdash^+ \Box A$$
$$\Rightarrow \quad \forall w'(wRw' \Rightarrow w' \Vdash^+ A)$$

$$\Pi_w \nvdash A \quad \Rightarrow \quad w \nVdash^+ \Box A$$
$$\Rightarrow \quad \exists w'(wRw' \text{ and } w' \nVdash^+ A)$$

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- They provide a representation in $\Im$ of provability in Π_w.

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- Hence it is not provable.
- But we have just proved this; so it is.

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- 'It is provable that A' can be false, even if is true as well.

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- Validity is defined as truth preservation over all *possible* worlds (worlds in P).

$\rightarrow$

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- If $w \in P$:
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- $\models A \vee \neg A$

Relevance

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Relevance

- $\rightarrow$ is a detachable conditional
- The logic is a relevant logic.

$\Box$

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- $\models \Box A \rightarrow \Box\Box A$
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- $\times \quad \models \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- $\Box(A \rightarrow B), \Box A \models \Box B$

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- Π_w is a theory
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- if G is true and false at all possible worlds in an interpretation, so are $\Box G$ and $\Box\neg G$.

Orlov's Project for Dialethesim

- The logic is a relevant logic
- It has Excluded Middle
- $\Box$ is a provability operator
- We may have both $\Box A$ and $\Box\neg A$

Conclusion

The End

Ivan Orlov

The End

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In a possible world where his paper had not fallen into oblivion... ?