Reflection algebras for theories of iterated truth predicates

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- Joint work with *Fedor Pakhomov*.
- Influenced by:
  - Feferman and Schütte's analysis of predicativity;
  - Ulf Schmerl's fine structure theorems for iterated reflection principles;
  - Kotlarski's et al. study of inductive satisfaction classes.
- Applies provability logic methods in mainstream proof theory (ordinal anaysis).

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Prehistory: Turing

G2: If a Gödelian T is consistent then $T \nvdash \mathrm{Con}(T)$.
A natural response: add $\mathrm{Con}(T)$ to T as a new axiom.
Is $T + \mathrm{Con}(T)$ complete? No, because it is Gödelian.

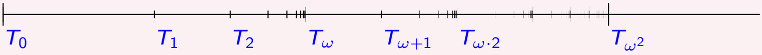
A. Turing (1939) suggested to continue the process:

$T_0 = T$
$T_1 = T + \mathrm{Con}(T)$
$T_2 = T + \mathrm{Con}(T) + \mathrm{Con}(T + \mathrm{Con}(T))$
...
$T_{n+1} = T_n + \mathrm{Con}(T_n)$
...

Is $\cup_{n \geq 0} T_n$ complete?

No: $T_\omega := \cup_{n\geq 0} T_n$ is Gödelian. Hence, T_ω does not prove $\mathrm{Con}(T_\omega)$ and the process continues:

$$T_{\omega+1} = T_\omega + \mathrm{Con}(T_\omega)$$
$$T_{\omega+2} = T_{\omega+1} + \mathrm{Con}(T_{\omega+1})$$
$$\ldots$$



Turing's classification program

Turing hoped to obtain a classification of all true arithmetical statements according to the stages of this (and similar) processes – but encountered difficulties.

A.M. Turing 1939 *System of logics based on ordinals*:

> *We might also expect to obtain an interesting classification of number-theoretic theorems according to "depth". A theorem which required an ordinal α to prove it would be deeper than one which could be proved by the use of an ordinal β less than α. However, this presupposes more than is justified.*

Canonical ordinal notations problem

Orderings can be represented in T, for example, by assigning rational numbers to points. The resulting set of numbers must be recognizable by an algorithm. (Otherwise, the axioms of T_α would not be recognizable.)

A problem: theories T_α depend on a particular way the ordering is computed rather than on the isomorphism type (the ordinal) of α.

Turing, Feferman, Kreisel: the whole classification idea breaks down because of this problem.

A partial way out: use canonical ordinal notations. But what exactly is a canonical ordinal notation system? Where do we get these notations from?

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Reflection principles and conservativity

Issue: There are Π^0_2-sentences not provable at any stage of a Turing progression T_α.

- *Reflection principles* $R_n(T)$ for T are arithmetical sentences expressing *"every Σ_n-sentence provable in T is true"*.

$R_n(T)$ is expressible as a Π_{n+1}-sentence, $R_0(T) := \mathrm{Con}(T)$.

- T is Π_{n+1}-*conservative* over U if U proves all Π_{n+1}-theorems of T. Denoted $T \subseteq_n U$.

We write $T \equiv_n U$ if both $T \subseteq_n U$ and $U \subseteq_n T$.
The bigger n, the "closer" T and U.

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Prehistory: Ulf Schmerl

Let EA be a basic *elementary arithmetic*.

The best we can hope with a Turing progression $R_n^\alpha(\mathsf{EA})$ is to approximate a given theory U from below up to $\equiv_n$. If U is *natural* it is likely that *natural* ordinal notations would suffice and at some point α we would obtain:

$$U \equiv_n R_n^\alpha(\mathsf{EA}).$$

Theorem (U. Schmerl, 1979): For all n,

$$\mathsf{PA} \equiv_n R_n^{\varepsilon_0}(\mathsf{EA}).$$

This result immediately yields: a consistency proof by transfinite induction, a classification of provably total recursive functions, provable well-orderings, etc.

Conclusions

- Proof-theoretic ordinals by iterated reflection principles provide the finest of the existing ordinal classifications of arithmetical theories.
- Establishing relations such as

$$U \equiv_n R_n^\alpha(\mathsf{EA})$$

  yields all the standard proof-theoretic information about U:

  - Consistency strength $n = 0$;
  - Classification of provably recursive functions $n = 1$;
  - Bounds on provable transfinite induction (cf also Pakhomov and Walsh, 2018).

Why truth predicates?

- Truth predicates are tightly related to reflection principles and are convenient in our framework.
- Theories of iterated truth are mutually interpretable with various standard theories of predicative strength (ramified analysis, iterated Π^0_1-comprehension).
- The framework remains first order and many ingredients are preserved from the treatment of PA.

Languages with truth predicates

- $\mathcal{L}$ first order language extending that of PA by finitely many predicate letters
- $\mathcal{L}_\alpha := \mathcal{L} \cup \{\mathsf{T}_\beta : \beta < \alpha\}$ new unary predicates
  (An elementary ordering representing ordinals up to α induces a Gödel numbering of $\mathcal{L}_\alpha$.)
- $\mathsf{T}_\alpha(\ulcorner\varphi\urcorner)$ means *"φ is a true $\mathcal{L}_\alpha$-sentence"*.

Uniform Tarski biconditionals

- Axioms UTB_α in $\mathcal{L}_{\alpha+1}$:
  U1 $\forall \vec{x}\,(\varphi(\vec{x}) \leftrightarrow \mathsf{T}_\alpha(\ulcorner \varphi(\underline{\vec{x}}) \urcorner))$, for each $\varphi(\vec{x}) \in \mathcal{L}_\alpha$;
  U2 $\neg \mathsf{T}_\alpha(\underline{n})$, if n is not a G.n. of an $\mathcal{L}_\alpha$-sentence.
- $\mathsf{UTB}_{<\alpha} := \bigcup_{\beta<\alpha} \mathsf{UTB}_\beta$ in $\mathcal{L}_\alpha$

Fact. $\mathsf{UTB}_{<\alpha}$ conservatively extends $\mathsf{UTB}_{<\beta}$ if $\beta < \alpha$.

(A model of $\mathsf{UTB}_{<\beta}$ can be extended to a model of $\mathsf{UTB}_{<\alpha}$.)

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Formula classes

Arithmetical hierarchy:

- $\Delta_0^{\mathcal{L}} = \Pi_0^{\mathcal{L}} = \Sigma_0^{\mathcal{L}}$ closure of atomic $\mathcal{L}$-formulas under $\wedge$, $\neg$ and bounded quantifiers;
- $\Pi_{n+1}^{\mathcal{L}} := \forall\vec{x}\,\varphi$ where $\varphi \in \Sigma_n^{\mathcal{L}}$;
- $\Sigma_{n+1}^{\mathcal{L}} := \exists\vec{x}\,\varphi$ where $\varphi \in \Pi_n^{\mathcal{L}}$.

Formula classes

Hyperarithmetical hierarchy:

- $\Pi_\alpha := \Pi^{\mathcal{L}}_{1+n}$ if $\alpha = n < \omega$;
- $\Pi_\alpha := \Pi^{\mathcal{L}_{\beta+1}}_{n+1}$ if $\alpha = \omega(1+\beta) + n$;
- $\Pi_{<\lambda} := \bigcup_{\alpha<\lambda} \Pi_\alpha$ if $\lambda \in \mathsf{Lim}$.

Rem. Π_α-formulas define $\Pi_1(\mathbf{0}^{(\alpha)})$-sets in $\mathbb{N}$.

- $\Pi_{<\omega}$ arithmetical (in $\mathcal{L}$) sets;
- $\Pi_\omega = \Pi_1(\mathbf{0}^{(\omega)}) = \Pi_1(\mathsf{T}_0)$.

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Reflection principles

Let S be Gödelian and $S \vdash \mathsf{EA}$.
$\Box_S$ is the provability predicate for S.

- $R_\alpha(S) := \{\forall \vec{x}\, (\Box_S \varphi(\underline{\vec{x}}) \to \varphi(\vec{x})) : \varphi \in \Pi_\alpha\}$;
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Π_α-*conservativity*

We fix an elementary well-ordering $(\Lambda, <)$ and hence:

- language $\mathcal{L}_\Lambda$;
- formula classes Π_α, for all $\alpha < \omega(1 + \Lambda)$;
- basic theory of iterated Tarski biconditionals $IB := \mathsf{EA}^+ + \mathsf{UTB}_{<\Lambda}$ where $\mathsf{EA}^+ := I\Delta_0 + \mathsf{Supexp}$.

Π_α-conservativity:

Def. $S \equiv_\alpha U$ means S and U prove the same Π_α-sentences.

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Two basic conservation results

$R^0_\alpha(S) := S$; $R^{n+1}_\alpha(S) := R_\alpha(S + R^n_\alpha(S))$;
$R^\omega_\alpha(S) := \bigcup_{n>0} R^n_\alpha(S)$.

Theorem
Let U be $\Pi_{\alpha+1}$-axiomatized extension of IB and $S \vdash U$. Over U, $R_{\alpha+1}(S) \equiv_\alpha R^\omega_\alpha(S)$.

- This is a relativization of the *reduction property* in arithmetic with an almost identical proof using cut-elimination.
- A well-known particular case is the Parsons–Mints–Takeuti theorem on the Π^0_2-conservativity of $I\Sigma_1$ over PRA.

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Two basic conservation results

Theorem
Let $\lambda \in \mathrm{Lim}$. Then, over IB, $R_\lambda(S) \equiv_{<\lambda} R_{<\lambda}(S)$.

We build a local $\Pi_{<\lambda}$-preserving interpretation of $IB + R_\lambda(S)$ into $IB + R_{<\lambda}(S)$.

Cor. $\mathrm{IB} + \mathrm{RFN}_{\Pi_1(\mathrm{T})}(S)$ is conservative over $PA + \mathrm{RFN}(S)$ for arithmetical sentences.

Rem. Both conservation theorems are formalizable in EA^+.

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Kotlarski theorem

Compositional truth axioms CT:

- $\forall\varphi\,(At[\varphi] \to (T[\varphi] \leftrightarrow T_0[\varphi]))$;
- $\forall\varphi,\psi\,(T[\varphi \wedge \psi] \leftrightarrow (T[\varphi] \wedge T[\psi]))$;
- $\forall\varphi\,(T[\neg\varphi] \leftrightarrow \neg T[\varphi])$;
- $\forall\varphi\,(T[\forall x\, \varphi(x)] \leftrightarrow \forall x\, T[\varphi(\underline{x})])$.

Cor. (Kotlarski) $\mathsf{PA} + \mathsf{CT} + I\Delta_0(\mathsf{T})$ is conservative over $\mathsf{PA} + \mathsf{RFN}^\omega(\mathsf{PA})$.

Proof.

Kotlarski theory is contained in $\mathsf{EA} + \mathsf{UTB} + R_1(\mathsf{EA} + \mathsf{UTB})$. Then apply Theorems 1 and 2. □

Semilattice of Gödelian theories

Def. $\mathfrak{G}_{\mathsf{IB}}$ is the set of all Gödelian extensions of IB mod $=_{\mathsf{IB}}$.

$S \leq_{\mathsf{IB}} T \iff \mathsf{IB} \vdash \forall x\, (\Box_T(x) \to \Box_S(x))$;

$S =_{\mathsf{IB}} T \iff (S \leq_{\mathsf{IB}} T \text{ and } T \leq_{\mathsf{IB}} S)$.

Then $(\mathfrak{G}_{\mathsf{IB}}, \wedge_{\mathsf{IB}})$ is a lower semilattice with $S \wedge_{\mathsf{IB}} T := S \cup T$ (defined by the disjunction of the numerations of S and T)

- Each R_α acts on $\mathfrak{G}_{\mathsf{IB}}$: $S \longmapsto \mathsf{IB} + R_\alpha(S)$;
- $(\mathfrak{G}_{\mathsf{IB}}, \wedge_{\mathsf{IB}}, (R_\alpha)_{\alpha<\Lambda})$ semilattice with a family of monotone operators.

Fact. Over IB the schemata $R_\alpha(S)$ are finitely axiomatizable.

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Reflection calculus RC_Λ

Language: $A ::= \top \mid p \mid (A \wedge A) \mid \alpha A \quad$ for $\alpha < \Lambda$
Sequents: $A \vdash B$

RC_Λ rules:

1. $A \vdash A$; $\quad A \vdash \top$; $\quad$ if $A \vdash B$ and $B \vdash C$ then $A \vdash C$;
2. $A \wedge B \vdash A, B$; $\quad$ if $A \vdash B$ and $A \vdash C$ then $A \vdash B \wedge C$;
3. if $A \vdash B$ then $\alpha A \vdash \alpha B$; $\quad \alpha\alpha A \vdash \alpha A$;
4. $\alpha A \vdash \beta A$ for $\alpha > \beta$;
5. $\alpha A \wedge \beta B \vdash \alpha(A \wedge \beta B)$ for $\alpha > \beta$.

Ex. $3\top \wedge 23p \vdash 3(\top \wedge 23p) \vdash 323p$.

Soundness of RC_Λ

An arithmetical interpretation is a map from RC_Λ-formulas to $\mathfrak{G}_{\mathsf{IB}}$ satisfying: $\top^* = \top$; $(A \wedge B)^* = A^* \wedge_{\mathsf{IB}} B^*$; $(\alpha A)^* = R_\alpha(A^*)$.

Th. If $A \vdash_{\mathrm{RC}_\Lambda} B$ then $A^* \leq_{\mathsf{IB}} B^*$, for any interpretation $*$.

RC_Λ *as an ordinal notation system*

Define: $A <_\alpha B$ iff $B \vdash \alpha A$.

- $\mathbb{W}$ is the set of all variable-free RC_Λ formulas.
- $\mathbb{W}_\alpha$ is the restriction of $\mathbb{W}$ to the signature $\{\beta : \alpha \leq \beta < \Lambda\}$.

Facts.

1. Every $A \in \mathbb{W}$ is equivalent to a word (formula without $\wedge$);
2. $(\mathbb{W}_\alpha, <_\alpha)$ is a well-ordering modulo equivalence in RC_Λ;
3. Its order type can be characterized in terms of Veblen φ function.

Ex. The order type of $(\mathbb{W}, <_0)$ in $\mathrm{RC}_{\omega^\alpha}$ is $\varphi_\alpha(0)$.

Veblen functions

- $\varphi_0(\beta) := \omega^{1+\beta}$;
- $\varphi_{\alpha+1}(\beta) := \beta$-th fixed point of φ_α;
- $\varphi_\lambda(\beta) := \beta$-th simultaneous fixed point of $\{\varphi_\alpha : \alpha < \lambda\}$, if $\lambda \in \mathsf{Lim}$.
- $\Gamma_0 :=$ the least ordinal > 0 closed under $\varphi_\alpha(\beta)$.

Fact. The order types of elements of $\mathbb{W}_\alpha \setminus \{\top\}$ within $(\mathbb{W}_0, <_0)$ are enumerated by φ_α.

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Schmerl-type formulas

Recall that $\equiv_\alpha$ denotes conservativity w.r.t. Π_α.
A^*_S denotes the interpretation of A in $\mathfrak{G}_S$.

Theorem
Let S be a $\Pi_{\alpha+1}$-axiomatizable extension of IB. In $\mathfrak{G}_S$, for all $A \in \mathbb{W}_\alpha$,

$$A^*_S \equiv_\alpha R_\alpha^{o_\alpha(A)}(S).$$

Cor. For any ordinal notations $\alpha, \beta, \gamma < \Gamma_0$,
$R^\gamma_{\alpha+\omega^\beta}(S) \equiv_\alpha R_\alpha^{\varphi_\beta(\gamma)}(S)$.

This holds, because $o_\alpha(A) = \varphi_\beta(o_{\alpha+\omega^\beta}(A))$ for $A \in \mathbb{W}_{\alpha+\omega^\beta}$.

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A few examples

1. Peano arithmetic: $\mathsf{PA} \equiv_{\Pi^0_{n+1}} R_n^{\varepsilon_0}(\mathsf{EA}^+)$.
2. $\mathsf{ACA} := \mathsf{PA}$ + arithmetical comprehension + full induction.
   Well-known: $\mathsf{ACA} \equiv \mathsf{PA}(\mathsf{T}_0) \equiv \mathsf{IB} + R_{<\omega 2}(\mathsf{IB})$.

   $\mathsf{ACA} \equiv_\omega \mathsf{IB} + R_\omega^{\varepsilon_0}(\mathsf{IB}) \equiv_{<\omega} \mathsf{IB} + R_{<\omega}^{\varepsilon_0}(\mathsf{IB})$;

   $\mathsf{ACA} \equiv_n \mathsf{IB} + R_n^{\varepsilon_{\varepsilon_0}}(\mathsf{IB})$ for $n < \omega$.
3. $\mathsf{ACA}^+ := \mathsf{ACA} + \forall X\, \exists Y\; Y = X^{(\omega)}$. Then
   $\mathsf{ACA}^+ \equiv \mathsf{PA}(\mathsf{T}_0, \mathsf{T}_1, \ldots, \mathsf{T}_\omega) \equiv \mathsf{IB} + R_{<\omega^2+\omega}(\mathsf{IB})$.
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   Hence, $\mathsf{ACA}^+ \equiv_\omega \mathsf{IB} + R_\omega^{\varphi_2(\varepsilon_0)}(\mathsf{IB})$.

Iterated arithmetical comprehension

Th.

1. $(\Pi^0_1\text{–CA}_0)_{\omega^\alpha} \equiv_{<\omega^{\alpha+1}} \mathsf{IB} + R_{<\omega^{\alpha+1}}(\mathsf{IB})$;
2. $(\Pi^0_1\text{–CA})_{\omega^\alpha} \equiv_{<\omega^{\alpha+1}+\omega} \mathsf{IB} + R_{<\omega^{\alpha+1}+\omega}(\mathsf{IB})$.

Th.

1. $(\Pi^0_1\text{–CA}_0)_{\omega^\alpha} \equiv_{<\omega} \mathsf{IB} + R^{\varphi_{\alpha+1}(0)}_{<\omega}(\mathsf{IB})$;
2. $(\Pi^0_1\text{–CA})_{\omega^\alpha} \equiv_{<\omega} \mathsf{IB} + R^{\varphi_{\alpha+1}(\varepsilon_0)}_{<\omega}(\mathsf{IB})$.

Iterated arithmetical comprehension

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1. $(\Pi^0_1\text{–CA}_0)_{\omega^\alpha} \equiv_{<\omega^{\alpha+1}} \mathsf{IB} + R_{<\omega^{\alpha+1}}(\mathsf{IB})$;
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Th.

1. $(\Pi^0_1\text{–CA}_0)_{\omega^\alpha} \equiv_{<\omega} \mathsf{IB} + R^{\varphi_{\alpha+1}(0)}_{<\omega}(\mathsf{IB})$;
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